Chapter 16B: Surface Integrals

The surface S:

Parametrization of the surface S: $x = x(u,v) \quad y = y(u,v) \quad z = z(u,v)$

Vector function of a surface S: $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$

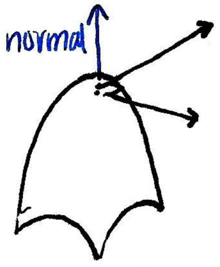
Area of a surface S: $A = \iint_S dS = \iint_{uv} |\mathbf{r}_u \times \mathbf{r}_v| dA$

Mass of a surface S: $mass = \iint_S f(x,y,z)dS = \iint_{uv} f(x(u,v), y(u,v), z(u,v))|\mathbf{r}_u \times \mathbf{r}_v| dA$

(with density $f(x,y,z)$)

Flux of **F** through a surface S: $\Phi = \iint_S \mathbf{F} \cdot \hat{n} dS = \iint_{uv} \mathbf{F}(x(u,v), y(u,v), z(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) dA$

I. What are two ways to get a normal vector to the surface $z = f(x, y)$ at point P?



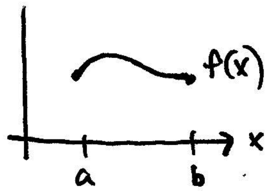
$z = f(x,y)$

1. Cross product of two vectors tangent to z at P

2. Let $g(x,y,z) = z - f(x,y)$
   $\rightarrow \nabla g|_P = \langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \rangle$

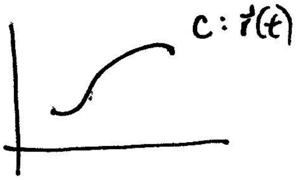
II. With Respect To … (w.r.t)

A. Area ⟶ $A = \int_a^b f(x)dx$ ⟶ infinite Riemann sum



$f(x)$ = sum-able quantity contributing to area
dx = infinite increment of change

B. Arc Length ⟶ $s = \int_C ds = \int_C \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$

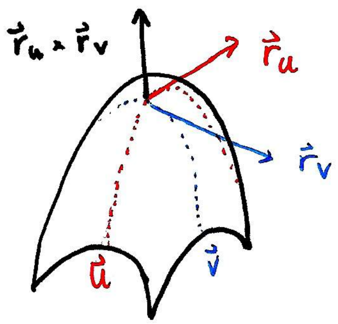


$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$ = sum-able quantity contributing to length
dt = infinite increment of change

C. Surface Area ⟶ $SA = \iint dS = \iint \underbrace{|\vec{r}_u \times \vec{r}_v|}_{\text{summable area patches}} \underbrace{du\,dv}_{dA}$ ⟶ parallelogram patches

summable area patches ↓ summable quantity

dA ↓ increment of change



$S: \vec{r}(u,v)$

* surfaces need two increments

* $\vec{r}_u \times \vec{r}_v \perp$ to plane containing $\vec{r}_u \times \vec{r}_v$

III. Find the parametrization $\mathbf{r}(u,v)$ (of the surface S)

A. The surface is a function:

Example: $z = 2x^2 + y^2 \quad \vec{r}(u,v) = \vec{r}(x,y) = \langle x, y, 2x^2 + y^2 \rangle$

General: $z = f(x,y) \rightarrow \vec{r}(x,y) = \langle x, y, f(x,y) \rangle$

$x = f(y,z) \rightarrow \vec{r}(y,z) = \langle f(y,z), y, z \rangle$

B. The surface is a circular cylinder:

Example: $x^2 + y^2 = 9 \quad \vec{r}(u,v) = \vec{r}(t,z) = \langle 3\cos t, 3\sin t, z \rangle$

General: $\left.\begin{array}{l} x^2 + y^2 = a^2 \quad \vec{r}(t,z) = \langle a\cos t, a\sin t, z \rangle \\ x^2 + z^2 = a^2 \quad \vec{r}(t,y) = \langle a\cos t, y, a\sin t \rangle \end{array}\right\} \quad \begin{array}{l} 0 \le t \le 2\pi \\ z \in \mathbb{R} \end{array}$

C. The surface is a sphere:

Example: $x^2 + y^2 + z^2 = 16$

General: $x^2 + y^2 + z^2 = a^2 \rightarrow \vec{r}(u,v) = \vec{r}(\phi, \theta) = \langle a\cos\theta \sin\phi, a\sin\theta\sin\phi, a\cos\phi \rangle$

IV. Find $(\mathbf{r}_u \times \mathbf{r}_v)$ and $|\mathbf{r}_u \times \mathbf{r}_v|$

A. The surface is a function: ex: let $z = f(x,y) \quad \vec{r}(x,y) = \langle x, y, f(x,y) \rangle$

$\vec{r}_u \times \vec{r}_v = \vec{r}_x \times \vec{r}_y$

$\vec{r}_x = \langle 1, 0, f_x \rangle$

$\vec{r}_y = \langle 0, 1, f_y \rangle$

$\vec{r}_x \times \vec{r}_y = \langle -f_x, -f_y, 1 \rangle \rightarrow |\vec{r}_x \times \vec{r}_y| = \sqrt{f_x^2 + f_y^2 + 1}$

$\boxed{\text{OR}}$ let $g(x,y,z) = z - f(x,y)$

$\nabla g = \langle -f_x, -f_y, 1 \rangle = \vec{r}_x \times \vec{r}_y$

always works for functions
doesn't work all the time

B. The surface is a circular cylinder (therefore the radius a is fixed):

$\vec{r}_t = \langle -a\sin t, a\cos t, 0 \rangle$

$\vec{r}_z = \langle 0, 0, 1 \rangle$

$\vec{r}_t \times \vec{r}_z = \langle a\cos t, a\sin t, 0 \rangle \rightarrow |\vec{r}_t \times \vec{r}_z| = \sqrt{a^2\cos^2 t + a^2 \sin t} = a$

fixed radius a
(somewhat like Jacobian)

C. The surface is a sphere (therefore the radius a is fixed):

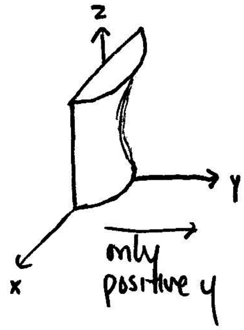
$\vec{r}_\phi = \langle a\cos\theta\cos\phi, \ldots, \ldots \rangle$

$\vec{r}_\theta = \langle -a\sin\theta\sin\phi, \ldots, 0 \rangle$

$\vec{r}_\phi \times \vec{r}_\theta = \langle \text{long} \rangle \rightarrow |\vec{r}_\phi \times \vec{r}_\theta| = a^2 \sin\phi$

$$SA = \iint dS = \iint_A |\vec{r}_u \times \vec{r}_v|\, dA \quad or \quad = \iint_A \nabla g\, dA$$

16.6#21. Find the parametric representation for the part of the hyperboloid $x^2 + y^2 - z^2 = 1$ that lies to the right of the xz-plane.



$$y = \pm\sqrt{1 - x^2 + z^2}$$

$$f(x,z) = \sqrt{1 - x^2 + z^2} \quad \text{with } 1 - x^2 + z^2 \geq 0$$

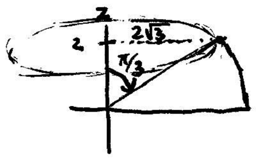
$$\vec{r}(x,z) = \langle x, \sqrt{1 - x^2 + z^2}, z \rangle$$

16.6 #24. Find the parametric representation for the part of the sphere $x^2 + y^2 + z^2 = 16$ that lies between the planes $z = -2$ and $z = 2$.

Figure: sphere chopped off

$$\vec{r}(\phi, \theta) = \langle 4\cos\theta\sin\phi,\ 4\sin\theta\sin\phi,\ 4\cos\phi \rangle \quad \text{with } \pi/3 \leq \phi \leq 2\pi/3$$

$$\text{and } 0 \leq \theta \leq 2\pi$$



$$x^2 + y^2 + 2^2 = 16$$

$$x^2 + y^2 = 12$$

16.6 #38 Find the area of the part of the plane $2x + 5y + z = 10$ that lies inside the cylinder $x^2 + y^2 = 9$.

$$z = 10 - 2x - 5y$$

$$f(x,y) = 10 - 2x - 5y$$

$$\vec{r}(x,y) = \langle x, y, 10 - 2x - 5y \rangle \quad x^2 + y^2 \leq 9$$

← use this if using $|\vec{r}_x \times \vec{r}_y|$ method

$\underline{\nabla g \text{ method}}$: $g = z - 10 + 2x + 5y$

$$\nabla g = \langle 2, 5, 1 \rangle$$

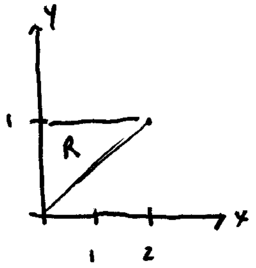
$$|\nabla g| = \sqrt{30}$$

Area: $A = \iint_{xy} |\vec{r}_x \times \vec{r}_y|\, dA = \iint_{xy} |\nabla g|\, dA$

$$= \iint_{xy} \sqrt{30}\, dA = \sqrt{30} \int_0^{2\pi}\int_0^{3} r\, dr\, d\theta = 9\sqrt{30}\,\pi$$

$$\text{OR} = \sqrt{30} \underbrace{\iint_{xy} dA}_{\text{area of circle} \to \text{projected onto } x\text{-}y} = \sqrt{30} \cdot \pi \cdot 3^2 = 9\sqrt{30}\,\pi$$

16.6 #42 Find the area of the part of the surface $z = 1 + 3x + 2y^2$ that lies above the triangle with vertices (0, 0), (0, 1), (2, 1).



$\vec{r}(x,y) = \langle x, y, 1 + 3x + 2y^2 \rangle$ where $1 + 3x + 2y^2 \in R$

∇g method : $g = z - 1 - 3x - 2y^2$

$\nabla g = \langle -3, -4y, 1 \rangle$

$|\nabla g| = \sqrt{16y^2 + 10}$

$$A = \iint_{xy} |\nabla g| \, dA = \int_0^2 \int_{x/2}^1 \sqrt{16y^2 + 10} \, dy \, dx$$

16.6 #44. Find the area of the part of the paraboloid $x = y^2 + z^2$ that lies inside the cylinder $y^2 + z^2 = 9$

$\vec{r}(y,z) = \langle y^2 + z^2, y, z \rangle \quad y^2 + z^2 \leq 9$

$g = x - y^2 - z^2$

$\nabla g = \langle 1, -2y, -2z \rangle$

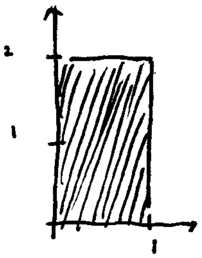
$|\nabla g| = \sqrt{1 + 4y^2 + 4z^2} = \sqrt{1 + 4(r^2)}$ let $y = \cos t \quad z = \sin t$

$$A = \iint_{xy} |\nabla g| \, dA = \int_0^{2\pi} \int_0^3 \sqrt{1 + 4r^2} \, r \, dr \, d\theta$$

let $u = 1 + 4r^2$
$du = 8r \, dr$

$$= \int_0^{2\pi} d\theta \cdot \int_1^{37} \sqrt{u} \, \frac{du}{8} = \frac{\pi}{4} \cdot \frac{2}{3} \left(37^{3/2} - 1^{3/2} \right) = \frac{\pi}{6} \left(37^{3/2} - 1 \right)$$

16.7-1: Evaluate $\iint\limits_S y\,dS$ where S is the surface $z = x + y^2$, $0 \le x \le 1$, $0 \le y \le 2$



$g = z - x - y^2$

$\nabla g = \langle -1, -2y, 1 \rangle$

$|\nabla g| = \sqrt{2 + 4y^2}$

$$\iint\limits_S y\,dS = \iint\limits_{xy} y\,|\vec{r}_x \times \vec{r}_y|\,dA = \int_0^1\!\!\int_0^2 y\,|\nabla g|\,dy\,dx$$

$$= \int_0^1\!\!\int_0^2 y\sqrt{2+4y^2}\,dy\,dx = \int_0^1 dx \int_0^2 y\sqrt{2+4y^2}\,dy$$

let $u = 2 + 4y^2$
$du = 8y\,dy$

$$= (1-0)\cdot\frac{1}{8}\int_2^{18}\sqrt{u}\,du = \frac{1}{_4\cancel{8}}\cdot\frac{\cancel{2}}{3}\left(18^{3/2} - 2^{3/2}\right) = \frac{13\sqrt{2}}{3}$$

16.7-2: Evaluate $\iint_S x^2 dS$ where S is the unit sphere $x^2 + y^2 + z^2 = 4$.

could be closed: $\oiint$

$\vec{r}(\phi, \theta) = \langle 2\cos\theta \sin\phi,\ 2\sin\theta \sin\phi,\ 2\cos\phi \rangle$ $\quad 0 \le \phi \le \pi$, $\quad 0 \le \theta \le 2\pi$

$|\vec{r}_\phi \times \vec{r}_\theta| = \cancel{4\sin^2\theta}$ $4\sin\theta$

$\iint_S x^2 ds = \iint_{\phi\theta} (4\cos^2\theta \sin^2\phi) \cdot \overbrace{\cancel{4\sin^2\theta}}^{4\sin\theta}\, d\theta\, d\phi$

$= 16 \int_0^{2\pi}\int_0^{\pi} \cancel{\sin^4\theta}\ \sin^3\theta \cos^2\phi\, d\phi\, d\theta = \frac{64\pi}{3}$

No Jacobian, because we didn't actually make a transformation

Correct Answer

16.7-1: Evaluate $\iint_S z\,dS$ where S is the surface whose sides S1 are given by $x^2 + y^2 = 1$, whose bottom S2 is the disk $x^2 + y^2 \le 1$ in the plane z = 0, and whose top S3 is the part of the plane $z = 1 + x$ that lies above S2.

S1 : $x^2+y^2=1$

$\vec{r}(t,z) = \langle \cos t, \sin t, z \rangle$

$|\vec{r}_t \times \vec{r}_z| = 1$

$$\iint_{S_1} z\,dS = \iint_{S_1} z \cdot 1\,dz\,dt = \int_0^{2\pi}\int_0^{(1+\cos t)} z\,dz\,dt = \frac{3\pi}{2}$$

S2 : $z=0 \quad x^2+y^2 \le 1$

$g = z$

$\nabla g = \langle 0,0,1 \rangle$

$|\nabla g| = 1$

$$\iint_{S_2} z\,dS = \iint_{S_2} 0 \cdot 1 \cdot dA = 0$$

S3 : $z = 1 + x$

$g = z - 1 - x$

$\nabla g = \langle -1, 0, 1 \rangle$

$|\nabla g| = \sqrt{2}$

$$\iint_{S_3} z\,dS = \iint_{xy} (1+x)\sqrt{2}\,dx\,dy$$

$$= \int_0^{2\pi}\int_0^{1} (1 + r\cos\theta)\sqrt{2}\,r\,dr\,d\theta = \sqrt{2}\pi$$

$$\iint_S z\,dS = \frac{3\pi}{2} + 0 + \sqrt{2}\pi = \left(\frac{3}{2} + \sqrt{2}\right)\pi$$

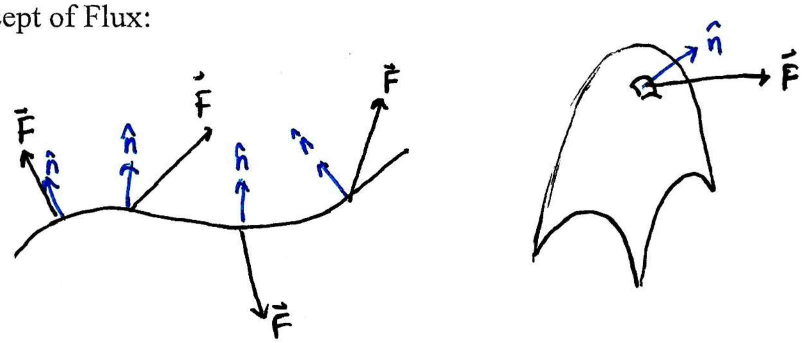
C3: Q204: CH16B LESSON2

FLUX and $\iint_S \mathbf{F} \cdot d\mathbf{S}$

Flux of **F** through a positive oriented surface S:

$$\Phi = \iint_S \mathbf{F} \cdot \hat{n} dS = \iint_{uv} \mathbf{F}(x(u,v), y(u,v), z(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) dA$$

Concept of Flux:



Contribution of $\vec{F}$ onto vector $\hat{n}$, summed :

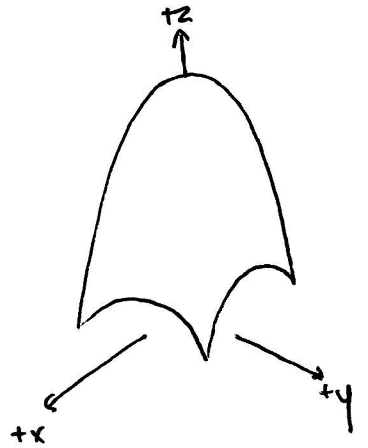
$$\iint_S \vec{F} \cdot \hat{n} \, dS = \iint \vec{F} \cdot \underbrace{\frac{(\vec{r}_u \times \vec{r}_v)}{|\vec{r}_u \times \vec{r}_v|}}_{\text{normal vector}} \underbrace{|\vec{r}_u \times \vec{r}_v| \, dA}_{dS} = \iint \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) \, dA = \iint \vec{F} \cdot d\vec{S}$$

OR

$$\iint \vec{F} \cdot \nabla g \, dA$$

(Concept) (Derivation) (Computation) (Notation)

If a surface can be described by $z = f(x, y)$ then the surface is considered positively oriented if the z-component of the normal vectors is positive. If a surfaces can be describes by $x = f(y, z)$ or $y = f(x, z)$, then the surface is considered positively oriented if the x component and y component, respectively, of the respective normal vectors are positive. If a surface is closed, then a positively oriented surface is one in which the normal vectors all point outward or away from the surface. For standardization, flux (or *flux-out*) implies through a positively oriented surface.



$z = f(x,y)$

Open surface

$z = f(x,y)$ with positive orientation = up $(+z)$

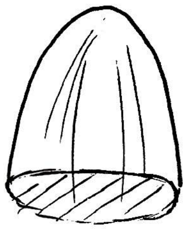
$\iint \vec{F} \cdot d\vec{S} > 0$ net flow in $+z$ (up)

$\iint \vec{F} \cdot d\vec{S} < 0$ net flow in $-z$ (up)

Ex: if $y = f(x,z)$ and positive = $+y$

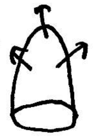
Flux positive net flow in $+y$ ("right")

Flux negative net flow in $-y$ ("left")



Closed surface

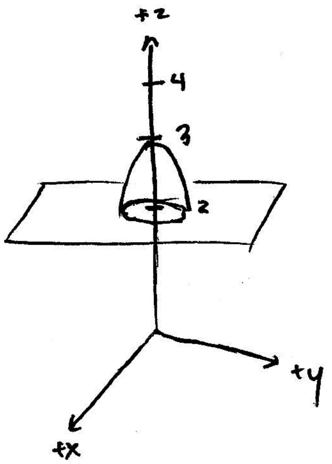
Positive orientation is outward (either up or down)

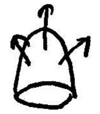
S_1  surface: pos is up

S_2  surface: pos is up

S  Flux (S) = Flux (S_1) + $-$Flux (S_2)

EXAMPLE 1: Find the flux of $\mathbf{F} = \langle y, x, z \rangle$ across the boundary of the solid region E enclosed by the paraboloid $z = 3 - x^2 - y^2$ and the plane $z = 2$.



S_1 : 

$$z = f(x,y) = 3 - x^2 - y^2$$

$$g = z - 3 + x^2 + y^2$$

$$\nabla g = \langle 2x, 2y, 1 \rangle$$

$$\vec{F}(x,y) = \langle y, x, 3 - x^2 - y^2 \rangle$$

$$\Phi_{S_1} = \iint_{xy} \vec{F} \cdot \nabla g \, dA = \iint_{xy} \left(2xy + 2xy + 3 - x^2 - y^2 \right) dA$$

$$= \int_0^{2\pi} \int_0^{1} \left(4r^2 \sin\theta \cos\theta + 3 - r^2 \right) r \, dr \, d\theta = \frac{5\pi}{2}$$

S_2 : 

$$z = 2 \quad x^2 + y^2 \leq 1$$

$$g = z - 2$$

$$\nabla g = \langle 0, 0, 1 \rangle$$

$$\vec{F}(x,y) = \langle y, x, 2 \rangle$$

$$\Phi_{S_2} = \iint_{xy} \vec{F} \cdot \nabla g \, dA = \iint_{xy} 2 \, dA = 2 \iint_{xy} dA = 2 \cdot \pi (1)^2 = 2\pi$$

$$\Phi = \Phi_{S_1} + \Phi_{S_2} = \frac{5\pi}{2} + \underbrace{(-2\pi)}_{} = \boxed{\frac{\pi}{2} \text{ gal/min}}$$

because "outward" (S) is also "downward" (S_2)

EXAMPLE 2: Find the flux of $\mathbf{F} = \langle z, y, x \rangle$ across the unit sphere $x^2 + y^2 + z^2 = 1$.

Parametrize: $\vec{r}(\phi, \theta) = \langle \cos\theta \sin\phi,\ \sin\theta \sin\phi,\ \cos\phi \rangle$

$$\vec{r}_\phi = \langle \cos\theta \cos\phi,\ \sin\theta \cos\phi,\ -\sin\phi \rangle$$

$$\vec{r}_\theta = \langle -\sin\theta \sin\phi,\ \cos\theta \sin\phi,\ 0 \rangle$$

$$\vec{r}_\phi \times \vec{r}_\theta = \langle \cos\theta \sin^2\phi,\ \sin\theta \sin^2\phi,\ \underbrace{\cos^2\theta \cos\phi \sin\phi + \sin^2\theta \cos\phi \sin\phi}_{\text{(Pythagorean)}} \rangle$$

$$= \langle \cos\theta \sin^2\phi,\ \sin\theta \sin^2\phi,\ \cos\phi \sin\phi \rangle$$

$$F = \langle \cos\phi,\ \sin\theta \sin\phi,\ \cos\theta \sin\phi \rangle$$

Calculate: $\iint \vec{F} \cdot d\vec{S} = \iint \vec{F} \cdot (\vec{r}_\phi \times \vec{r}_\theta)\, dA$

$$= \int_0^{2\pi}\int_0^{\pi} \left(\underbrace{2\cos\theta \cos\phi \sin^2\phi}_{0} + \underbrace{\sin^2\theta \sin^3\phi}_{\frac{4}{3}\pi} \right) d\phi\, d\theta$$

ORGANIZING FLUX (Bridging the Gap Between the Mathematical and Physical Understanding)
By Mr. Michael W. Bermel with supporting details by Mr. Sameer Jain.

In general terms, transport flux is a measure of the rate at which stuff flows through a surface. Although most authors do not distinguish between "point flux" and "total (or net) flux", I find it helpful to understand this difference. Let us define (1) *point flux* to be the rate, per unit of area, of stuff flowing through the surface at a point and (2) *total flux* to be the net rate of stuff flowing through the entire surface. There are several kinds of transport fluxes and each of these types defines the rate of stuff flowing differently.

	Units of Flux		
		(1) Point Flux Flow Rate per Unit of Area *Quantity / time / area*	(2) Total Flux Flow Rate across the total surface area: *Quantity / time*
Transport Flux	Volumetric Flux $\vec{F} = \vec{v}$	$\frac{m^3}{s} \Big/ m^2$ or simplified to *(distance/time)*	$\frac{m^3}{s} \Leftarrow (m^2)\left(\frac{m}{s}\right)$
	Mass Flux $[\rho] = \frac{kg}{m^3}$ $\vec{F} = \rho\vec{v}$	$\frac{kg}{s} \Big/ m^2 \Leftarrow \frac{kg}{m^3} \cdot \frac{m^3/s}{m^2} = \frac{kg/s}{m^2}$	$\frac{kg}{s} \Leftarrow \left(\frac{kg/s}{m^2}\right) m^2$
	Heat Flux * negative: H to L $\vec{F} = -K\nabla u$	$\frac{J}{s} \Big/ m^2$ or $Watts \big/ m^2 \Leftarrow \dfrac{\frac{W\cdot m}{K} \cdot \frac{K}{m}}{m^2}$	$\frac{J}{s}$ or *Watts*

(Other Transport Fluxes include: Momentum, Diffusion, Radiative, Energy, and Particle Flux)

Heat: $[K] = \frac{W\cdot m}{K}$ $[\nabla u] = K/m$
(temp wrt distance)

Flux is conceptually defined as $\Phi = \iint_S \mathbf{F}\cdot\hat{n}\,dS$ but is notated $\Phi = \iint_S \mathbf{F}\cdot d\mathbf{S}$

$\hat{n}$ is the unit vector normal to the surface S. $\hat{n} = \dfrac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$

dS is a small (infinite) patch of area. $dS = |\mathbf{r}_u \times \mathbf{r}_v|\,dA$

This equivalence is described here:

$$\Phi = \iint_S \mathbf{F}\cdot\hat{n}\,dS = \iint \mathbf{F}\cdot\frac{(\mathbf{r}_u \times \mathbf{r}_v)}{|\mathbf{r}_u \times \mathbf{r}_v|}|\mathbf{r}_u \times \mathbf{r}_v|\,dA = \iint_S \mathbf{F}\cdot(\mathbf{r}_u \times \mathbf{r}_v)\,dA = \iint_S \mathbf{F}\cdot d\mathbf{S}$$

For notational purposes $(\mathbf{r}_u \times \mathbf{r}_v)\,dA = d\mathbf{S}$

UNITS of NON – TRANSPORT FLUX

		(1) Point Flux Quantity / Area	(2) Total Flux Quantity
Electric Flux	Electric Flux	$\frac{N \cdot m^2}{C} \Big/ m^2$ *(Quantity/area)* Quantity/area = **Line Density**[1] = (charge/ ε_0 /area) Charge has units C (Coulombs) $\varepsilon_0 = 8.8542 \times 10^{-12} C^2 \cdot N^{-1} \cdot m^{-2}$ It is an accepted standard to use ε_0 (the permittivity of free space) in the calculation of **Line Density** and, hence, in electric flux. Note: (**Line Density**) has units (*N/C*) which could be thought of as Force per Charge.	$\frac{N \cdot m^2}{C}$ (*Quantity*)

For definition and details of **Line Density** see the excerpt of the paper written by Mr. Sameer Jain following this discussion.

[1] Definition of **Line Density** developed by Mr. Sameer Jain © 2015

Thoughts on Electromagnetic Flux
Sameer Jain
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In mechanics, volumetric, mass, and energy flux represent the rate **with respect to time** at which a physical quantity is being transported across a surface. Volume, mass, and energy are physical quantities inherent to all matter.

In electromagnetism, flux is a measure of the number of electromagnetic field lines through a given area. The vector fields that model electric and magnetic forces is **not time dependent**. Therefore, **electromagnetic flux does not measure a rate of change**.

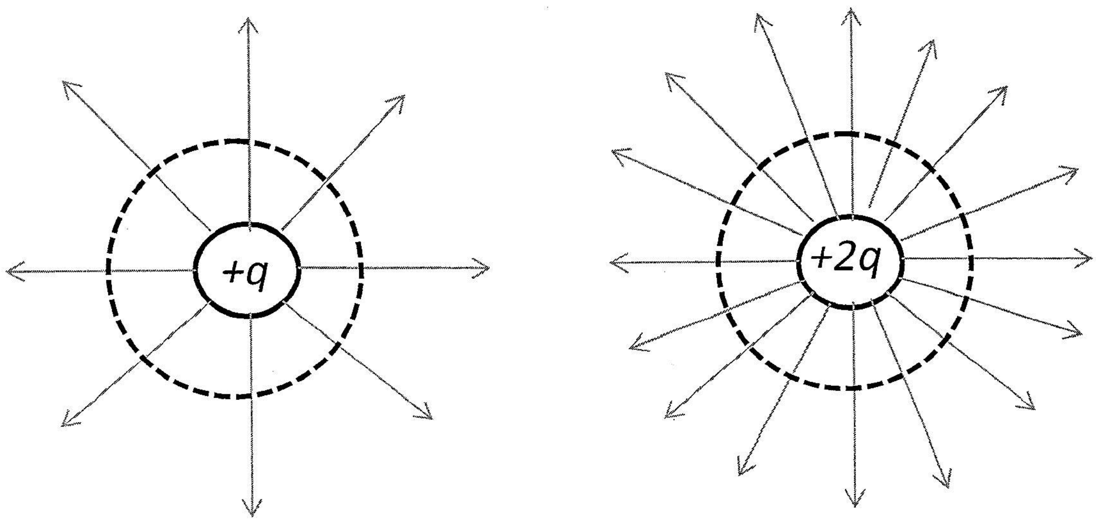
Physicists utilize electric flux to determine the **strength of electric fields**. The strength of an electric field is directly related to the flux measured through surfaces enclosing the source of the field. The source of an electric field is a charge.

Electricity is based on the fundamental principles that like charges repel and opposite charges attract. By convention, electric field lines are drawn outward from positive charges and towards negative charges. These field lines are modeled by mathematic vector fields that are dependent on the positions of test charges. In contrast, vector fields in mechanics are dependent on time.

PROOF: The density of electric field lines indicates the strength of the electric field.

Definition: *The density of electric field lines (line density) is defined as the number of field lines that cross a surface perpendicular to the lines divided by the area of that respective surface.*

We utilize a convention that for every coulomb of charge, we will draw a specific number of lines. Let us define that specific number of lines to be 8.



As seen in the figure, the density of electric field lines decreases in both cases as we move farther from the charge. Given the same distance away from the central charges, the density of electric field lines is substantially greater when the source of the electric field has a greater electric charge.

To create a convenient standard, physicists agree that for every one coulomb of charge, $\frac{1}{\epsilon_0}$ field lines will be drawn, where ϵ_0 is a constant defined as the permittivity of free space. Therefore a charge q would have $\frac{q}{\epsilon_0}$ electric field lines drawn radially outwards.

Utilizing the definition stated earlier, a charge q enclosed by a spherical surface (area = $4\pi r^2$) would have a line density defined below.

$$Line\ Density = \frac{number\ of\ field\ lines}{surface\ area} = \frac{\frac{q}{\epsilon_0}}{4\pi r^2} = \frac{q}{4\pi\epsilon_0 r^2} = |\vec{E}|$$

The derived density of electric field lines is identical to the **magnitude of the electric field** defined by physicists. The magnitude of the electric field (measured in N/C) is a representation of the strength of the electric field.

Therefore, the density of electric field lines indicates the strength of the electric field. The opposite also holds true. The strength (magnitude) of an electric field is equal to the density of the electric fields generated by the source of the electric field.

The magnitude of the electric field can be defined in two ways:

- The force acting on a coulomb of charge at a given position (N/C)
- The number of field lines per unit of area

When calculating the electric flux around a charge, unit analysis yields that the SI Units for electric flux are newton meters squared per coulomb (Nm^2C^{-1}). However, electric flux is never understood by these units.

Q204: Chapter 16B: Lesson 3 – The Divergence Theorem

Let E be a region in three dimensions bounded by a closed surface S, and let $\hat{n}$ denote the unit outer normal vector to S at (x, y, z). If $\mathbf{F}$ is a vector function that has continuous partial derivatives on E, then :

$$\iint_S \mathbf{F} \cdot \hat{n}\, dS = \iiint_E \underbrace{\nabla \cdot \mathbf{F}}_{\text{div } F}\, dV$$

where E is the 3D region enclosed

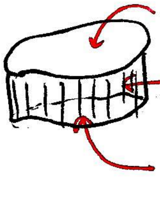
In other words, the flux of $\mathbf{F}$ over S equals the triple integral of the divergence of $\mathbf{F}$ over E.

Proof of Divergence Theorem

$$\iint F\cdot\hat{n}\,dS = \iint \langle P,Q,R\rangle\cdot\hat{n}\,dS$$
$$= \iint\langle P,0,0\rangle\hat{n}\,dS + \iint\langle 0,Q,0\rangle\hat{n}\,dS + \iint\langle 0,0,R\rangle\,dS$$
$$= \iiint\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dV$$

First prove $\displaystyle\iint_S\langle 0,0,R\rangle\cdot\hat{n}\,dS = \underbrace{\iint_{S_1} + \iint_{S_2} - \iint_{S_3}}_{\text{(shorthand)}} = \iiint\frac{\partial R}{\partial z}\,dV$

outwards flux



$S_1:\ z=u(x,y)$ Ⓑ

$S_3:\ \hat{n}=\langle \#,\#,0\rangle$ Ⓐ

$S_2:\ z=v(x,y)$ Ⓒ

Ⓐ $\displaystyle\iint_{S_3}\langle 0,0,R\rangle\cdot\hat{n}\,dS = \iint_{S_3}\langle 0,0,R\rangle\cdot\langle\#,\#,0\rangle\,dS = 0$

Ⓑ Find $\displaystyle\iint_{S_1}\langle 0,0,R\rangle\cdot\hat{n}\,dS$: let $g_1 = z-u(x,y) \rightarrow \nabla g_1 = \langle -u_x,-u_y,1\rangle$

$$\hat{n}=\frac{\nabla g_1}{|\nabla g_1|}=\frac{\langle -u_x,-u_y,1\rangle}{\sqrt{1+u_x^2+u_y^2}}\qquad dS=|\nabla g_1|\,dA=\sqrt{1+u_x^2+u_y^2}\,dA$$

$$\iint_{S_1}\langle 0,0,R\rangle\cdot\hat{n}\,dS = \iint_{S_1}R\cdot\frac{\nabla g_1}{|\nabla g_1|}\cdot|\nabla g_1|\,dA = \iint_{S_1}R\,dA = \iint_{S_1}(R(x,y,u(x,y)))\,dA$$

Ⓒ Find $\displaystyle\iint_{S_3}\langle 0,0,R\rangle\cdot\hat{n}\,dS$: let $g_2 = z-v(x,y)\rightarrow\nabla g_2=\langle -v_x,-v_y,1\rangle$

By a similar derivation, $\displaystyle\iint_{S_3}\langle 0,0,R\rangle\cdot\hat{n}\,dS = \iint_{S_3}R\,dA = \iint_{S_3}(R(x,y,v(x,y)))\,dA$

Sum: $\displaystyle\iint_S\langle 0,0,R\rangle\cdot\hat{n}\,dS = \iint_{S_1}\langle 0,0,R\rangle\cdot\hat{n}\,dS + \overbrace{\iint_{S_2}\langle 0,0,R\rangle\cdot\hat{n}\,dS}^{=0} \overset{\text{opp. orientation}}{-} \iint_{S_3}\langle 0,0,R\rangle\cdot\hat{n}\,dS$

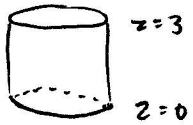
$$= \iint_D[R(x,y,u(x,y)) - R(x,y,v(x,y))]\,dA$$
$$= \iint_D\left[\int_{v(x,y)}^{u(x,y)}\frac{\partial R}{\partial z}\,dz\right]dA = \iiint\frac{\partial R}{\partial z}\,dV \quad \text{(FTC in reverse)}$$

Similar for $\displaystyle\iint_S\langle P,0,0\rangle\cdot\hat{n}\,dS = \iiint\frac{\partial P}{\partial x}\,dV$

$\displaystyle\iint_S\langle 0,Q,0\rangle\cdot\hat{n}\,dS = \iiint\frac{\partial Q}{\partial y}\,dV$

$$\therefore \oiint\vec{F}\cdot\hat{n}\,dS = \iiint\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dV$$
$$= \iiint\nabla\cdot\vec{F}\,dV \qquad \text{Q.E.D.}$$

1. Let E be the region bounded by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $z = 3$, and let S denote the surface of E. If $\mathbf{F} = \langle x^3, y^3, z^3 \rangle$, use the divergence theorem to find $\iint_S \mathbf{F} \cdot \hat{n} dS$.



$$\oiint_S \vec{F} \cdot \hat{n}\, dS = \iiint_E \text{div} F\, dV$$

$$= \iiint_{xyz} (3x^2 + 3y^2 + 3z^2)\, dV$$

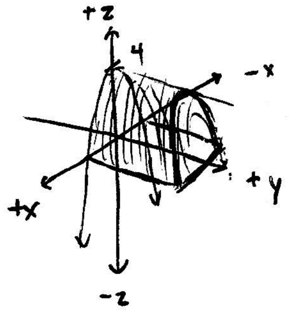
$$= 3 \int_0^{2\pi} \int_0^2 \int_0^3 (r^2 + z^2)\, r\, dz\, dr\, d\theta$$

$$= 3 \int_0^{2\pi} \int_0^2 \left[r^3 z + \frac{1}{3} z^3 \right]_0^3 dr\, d\theta = 3 \int_0^{2\pi} \int_0^2 (3r^3 + 9)\, dr\, d\theta$$

$$= 9 \int_0^{2\pi} d\theta \int_0^2 (r^3 + 3)\, dr$$

$$= 18\pi \cdot \left[\frac{1}{4} r^4 + 3r \right]_0^2 = 18\pi \cdot (4 + 6) = \boxed{180\pi}$$

2. Let E be the region bounded by the cylinder $z = 4 - x^2$, the plane $y + z = 5$ and the xy – and xz – planes, and let S be the surface of E. If $\mathbf{F} = \langle x^3 + \sin z, x^2 y + \cos z, e^{x^2+y^2} \rangle$, use the divergence theorem to find $\iint_S \mathbf{F} \cdot \hat{n} dS$.



$$\iint_S F \cdot \hat{n}\, dS = \iiint_{xyz} (3x^2 + x^2 + 0)\, dV = 4 \iiint_{xyz} x^2\, dV$$

$$= 4 \int_{-2}^{2} \int_{0}^{4-x^2} \int_{0}^{5-z} x^2\, dy\, dz\, dx = 4 \int_{-2}^{2} \int_{0}^{(4-x^2)} \left[x^2 y\right]_0^{5-z} dz\, dx$$

$$= 4 \int_{-2}^{2} \int_{0}^{(4-x^2)} \left(5x^2 - x^2 z\right) dz\, dx = 4 \int_{-2}^{2} \left[5x^2 z - \tfrac{1}{2}x^2 z^2\right]_0^{4-x^2} dx$$

$$= 4 \int_{-2}^{2} \left(5x^2(4-x^2) - \tfrac{1}{2}x^2(4-x^2)^2\right) dx$$

$12x^2 - x^4 - \frac{1}{2}x^6$

$$= 4 \int_{-2}^{2} \left(20x^2 - 5x^4 - 8x^2 - \tfrac{1}{2}x^6 + 4x^4\right) dx$$

$$= 4 \left[4x^3 - \tfrac{1}{5}x^5 - \tfrac{1}{14}x^7\right]_{-2}^{2}$$

$$= 4 \left(32 - \tfrac{32}{5} - \tfrac{64}{7}\right) - \left(-32 + \tfrac{32}{5} + \tfrac{64}{7}\right)$$

$$= 4 \cdot 2 \cdot \left(\frac{32 \cdot 35}{35} - \frac{32 \cdot 7}{35} - \frac{64 \cdot 5}{35}\right) = \boxed{\frac{4608}{35}}$$

CHALLENGE. It can be shown that the divergence of every inverse square field is zero. Now suppose a closed surface S forms the boundary of a region E and the origin O is an interior point of E. If an inverse square field is given by $\mathbf{F} = (q/r^3)\mathbf{r}$, where q is a constant, $\mathbf{r} = \langle x, y, z \rangle$, and $|\mathbf{r}| = r$, prove the flux of F over S is $4\pi q$ regardless of the shape of E.

$$\iint_S \vec{F}\cdot\hat{n}\, dS = \iiint_E \operatorname{div}\vec{F}\, dV = \iiint_E 0\, dV = 0 \quad (\underline{\text{Given}})$$

$$\text{Now: } \iint_S \vec{F}\cdot\hat{n}\, dS = \iint_{S_1} \vec{F}\cdot d\vec{S}_1 + \iint_{S_2(sys)} \vec{F}\cdot d\vec{S}_{2\,(sys)}$$

S_2 as part of system

$$= \iint_{S_1} \vec{F}\cdot d\vec{S}_1 - \iint_{S_2} \vec{F}\cdot d\vec{S}_2$$

S_2 independently

$$\text{Thus: } \iint_{S_1} \vec{F}\cdot d\vec{S}_1 = \iint_{S_2} \vec{F}\cdot d\vec{S}_2 = -\iint_{S_2(sys)} \vec{F}\cdot d\vec{S}_{2\,(sys)}$$

$$= -\iint_{S_2} \left(\frac{q}{r^3}\right)\vec{r}\cdot\left(-\frac{1}{r}\right)\vec{r}\, dS \qquad \hat{n} = \left(-\frac{1}{r}\right)\vec{r} \text{ (see below)}$$

$$= \iint_{S_2} \left(\frac{q}{r^4}\right)(\vec{r}\cdot\vec{r})\, dS = \iint_{S_2} \left(\frac{q}{r^4}\right)|\vec{r}|^2 dS$$

For $r = a$

$$= \iint_{S_2} \left(\frac{q}{r^2}\right) dS = \frac{q}{a^2}\iint_{S_2} dS = \frac{q}{\cancel{a^2}}(4\pi \cancel{a^2}) = 4\pi q$$

$$\therefore \iint_{S_1} \vec{F}\cdot d\vec{S}_1 = 4\pi q \quad \text{QED.}$$

Proof that $\hat{n} = -\frac{1}{r}\vec{r}$: for sphere with $r = a$

$$\vec{r}(x,y,z) = \vec{r}(\phi,\theta) = \langle a\cos\theta\sin\phi,\ a\sin\theta\sin\phi,\ a\cos\phi \rangle$$

$$\vec{r}_\phi \times \vec{r}_\theta = \langle a^2\cos\theta\sin^2\phi,\ a^2\sin\theta\sin^2\phi,\ a^2\cos\phi\sin\phi \rangle$$

$$\hat{n} = \frac{\vec{r}_\phi \times \vec{r}_\theta}{|\vec{r}_\phi \times \vec{r}_\theta|} = \frac{\text{schtuff}}{a^2\sin\phi} = \langle \cos\theta\sin\phi,\ \sin\theta\sin\phi,\ \cos\phi \rangle = \frac{\vec{r}}{a} = \frac{\vec{r}}{r}$$

Since flux through S_2 is inward, use $n = -\frac{\vec{r}}{r} = \left(-\frac{1}{r}\right)\vec{r}$ QED.

Q204: Chapter 16B: Lesson 4 – Stokes' Theorem

Introduction:

($\mathbb{R}^2$) $\vec{F} = \langle P, Q, 0 \rangle$ $\quad C: x = x(t) \quad y = y(t) \quad z = 0 \quad \rightarrow d\vec{r} = \langle dx, dy, 0 \rangle$

Review: $\oint_C \vec{F} \cdot \vec{T}\, ds = \oint_C \vec{F} \cdot d\vec{r} = \oint_C (P\,dx + Q\,dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_D \text{curl}\, \vec{F} \cdot \hat{k}\, dA$

Green's Theorem

$$\text{curl}\, \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

($\mathbb{R}^3$) $\vec{F} = \langle P, Q, R \rangle$ $\quad C: x = x(t) \quad y = y(t) \quad z = z(t) \quad \rightarrow d\vec{r} = \langle dx, dy, dz \rangle$

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C (P\,dx + Q\,dy + R\,dz) = \iint_S \text{curl}\, \vec{F} \cdot \hat{n}\, dS = \iint_S \text{curl}\, \vec{F} \cdot d\vec{S}$$

Stokes' Theorem

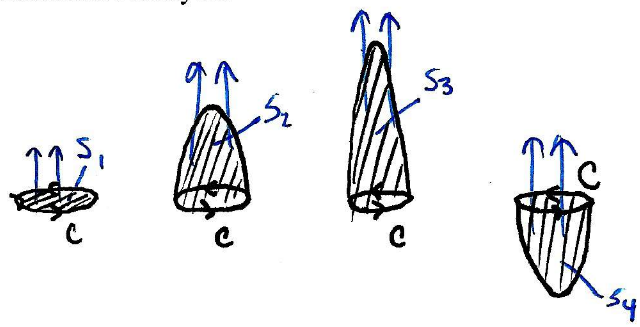
Stokes' Theorem:

$$\boxed{\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\, \vec{F} \cdot d\vec{S}}$$

$\rightarrow$ a general form of Green's Theorem

$\therefore$ Work done by $\vec{F}$ as particle moves along closed curve C equals flux of curl $\vec{F}$ over surface S, with C as boundary

Additional Analysis



$$\iint_{S_1} \text{curl}\,\vec{F}\cdot dS_1 = \iint_{S_2} \text{curl}\,\vec{F}\cdot dS_2 = \iint_{S_4} \text{curl}\,\vec{F}\cdot dS_4 = \oint_C \vec{F}\cdot d\vec{r}$$

because it's verify, it matters what curve / surface

1. Let S be the part of the paraboloid $z = 9 - x^2 - y^2$ with $z \geq 0$, and let C be the trace of S on the xy-plane. $\boxed{\text{Verify}}$ Stokes' Theorem for the vector field $\mathbf{F} = \langle 3z, 4x, 2y \rangle$.

$S: z = 9 - x^2 - y^2 \quad z \geq 0$

$C:$ xy trace of S

Line integral: $\oint \vec{F} \cdot d\vec{r}$

$x(t) = 3\cos t \quad y(t) = 3\sin t \quad z(t) = 0 \quad 0 \leq t \leq 2\pi$

$\vec{r}(t) = \langle 3\cos t, 3\sin t, 0 \rangle \quad 0 \leq t \leq 2\pi$

$d\vec{r} = \langle -3\sin t, 3\cos t, 0 \rangle \, dt$

$\vec{F}(t) = \langle 0, 12\cos t, 6\sin t \rangle$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} 36\cos^2 t \, dt = 36 \underbrace{\int_0^{2\pi} \cos^2 t \, dt}_{\pi} = 36\pi$$

memorize

Surface integral: $\iint \text{curl}\, \vec{F} \cdot d\vec{S} = \iint \text{curl}\, \vec{F} \cdot \nabla g \, dA$

$g = z - 9 + x^2 + y^2 \quad \nabla g = \langle 2x, 2y, 1 \rangle$

$$\text{curl}\, \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3z & 4x & 2y \end{vmatrix} = \langle 2, 3, 4 \rangle$$

$$\iint \text{curl}\, \vec{F} \cdot d\vec{S} = \iint_{xy} (4x + 6y + 4) \, dA$$

$$= \int_0^{2\pi}\int_0^3 (4r\cos\theta + 6r\sin\theta + 4) \, r \, dr \, d\theta = \cdots = 36\pi$$

If question asked for "solve": surface cheat (use simpler surface)

→ Just use a disk $z = 0 \quad x^2 + y^2 \leq 9$

$g = z \quad \nabla g = \langle 0, 0, 1 \rangle \quad \text{curl}\, \vec{F} = \langle 2, 3, 4 \rangle$

$$\iint \text{curl}\, \vec{F} \cdot d\vec{S} = \iint \text{curl}\, \vec{F} \cdot \nabla g \, dA = \iint_{xy} 4 \, dA = 4 \cdot \pi \cdot 3^2 = 36\pi$$

2. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = \langle -y^2, x, z^2\rangle$ and C is the curve of intersection of the plane $y+z=2$ and the cylinder $x^2+y^2=1$. (Orient C to be counterclockwise when viewed from above.)

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,dS = \iint_S \text{curl}\,\vec{F}\cdot\nabla g\,dA$$

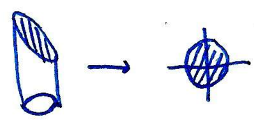
S: $y+z=2 \quad x^2+y^2\le 1$

$g = y+z-2 \quad \nabla g = \langle 0,1,1\rangle \quad \vec{F} = \langle -y^2, x, z^2\rangle$

$$\text{curl}\,\vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = \langle 0,0,1+2y\rangle$$

$$\text{curl}\,\vec{F}\cdot\nabla g = 1+2y$$

$$\iint \text{curl}\,\vec{F}\cdot d\vec{S} = \iint_{xy} (1+2y)\,dA = \underbrace{\int_0^{2\pi}\int_0^1}_{\text{Domain}} (1+2r\sin\theta)\,r\,dr\,d\theta = \cdots = \pi$$



Integrating over projected region